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Penrose limits of homogeneous spaces

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Abstract

We prove that the Penrose limit of a spacetime along a homogeneous geodesic is a homogeneous plane wave spacetime and that the Penrose limit of a reductive homogeneous spacetime along a homogeneous geodesic is a Cahen–Wallach space. We then consider several homogenous examples to show that these results are indeed sharp and conclude with a remark about the existence of null homogeneous geodesics.

MSC: 83C35; 53C30

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1. Introduction

In [25] Penrose introduced a method for taking a continuous limit of any spacetime to a plane wave. The method effectively involves “zooming in" on a null geodesic in such a way that the metric stays nondegenerate. In [14] Güven extended the method to that of supergravity theories where it is a useful tool for generating new solutions to the supergravity equations from known ones. Since then several papers have investigated the properties of these Penrose limits, [3–6,24].

Penrose limits have been used as evidence for the AdS/CFT correspondence. The Penrose limits of the $AdS_5 \times S^5$ type IIB superstring background were calculated in [5], one of which was shown to be the BFHP maximally supersymmetric plane wave background [4]. String theory in this background is exactly solvable [21,22] giving rise to an explicit form of the AdS/CFT correspondence [2] in which both the gauge theory and the gravity sides are weakly coupled, allowing many perturbative checks albeit for a restricted class of observables.

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A more general class of background metrics on which string theory is exactly solvable are the homogeneous plane waves [8,23]. Penrose limits onto homogeneous plane waves have been investigated, such as the Penrose limits of the Gödel-like spacetimes [6]. In [5] it was shown that the dimension of the isometry algebra never decreases under a Penrose limit. Hence it seemed a “natural" assumption that the Penrose limit of a homogeneous spacetime is always a homogeneous plane wave. However, in [24] it was shown that the cross product of the homogeneous Kaigorodov spacetime with a sphere has a Penrose limit which is not itself homogeneous. Consequently the aim of this paper is give necessary and sufficient conditions on a spacetime and the null geodesic that guarantee that the Penrose limit is homogeneous.

Section 2 gives the definition of the Penrose limit and proves that this definition is well–defined. We also give a proof of the covariance property of the Penrose limit stated in [5].

Section 3 gives some examples of known hereditary properties of the Penrose limit. Section 4 contains the background on homogeneous spaces needed for our results. This includes descriptions of reductive homogeneous spaces, naturally reductive homogeneous spaces, the Killing transport and homogeneous geodesics.

In Section 5 we use the Killing transport to prove that the Penrose limit of a lorentzian spacetime along a homogeneous geodesic is a homogeneous plane wave. We then use a similar approach to prove that the Penrose limit of a reductive homogeneous spacetime along a homogeneous geodesic is a reductive homogeneous plane wave.

In Section 6 we use the classification of homogeneous plane waves that was given in [7] to prove that the Penrose limit of a reductive homogeneous spacetime along an absolutely homogeneous geodesic is a naturally reductive plane wave.

In Section 7 we first show that the Penrose limit of a non–homogeneous spacetime can be homogeneous. Then we describe the Kaigorodov spacetime and its Penrose limits as calculated in [24]. We give an example of a Penrose limit of a reductive homogeneous space along a homogeneous geodesic for which the homogeneous structure “blows up” but the limiting spacetime is still homogeneous. We also give an example of a Penrose limit of a non–reductive homogeneous spacetime along a homogeneous geodesic which is still non–reductive homogeneous.

Finally in Section 8 we show that while there must exist at least one homogeneous geodesic in any reductive homogeneous spacetime [19,18], there may not exist any null absolutely homogeneous geodesics.

2. What is a Penrose limit?

Let (M, g) be a smooth $(n+1)$-manifold with a lorentzian metric. Let γ be a null geodesic of (M, g). Then given a point $x \in \gamma$ there exists a coordinate neighborhood (U, μ), $\mu : U \to \mathbb{R}^{n+1}$, of x defining coordinates $\mu(y) = (u(y), v(y), [y^k(y)])$, where u is a coordinate along γ, such that in U the metric may be written as

$$g = \mathrm{d}u\mathrm{d}v + \alpha \mathrm{d}v^2 + \sum_{i=1}^{n-1} \beta_i \mathrm{d}y^i \mathrm{d}v + \sum_{i,j=1}^{n-1} C_{ij} \mathrm{d}y^i \mathrm{d}y^j. \tag{1}$$

Here α, β_i, C_{ij} are functions of $(u, v, [y^k])$ and (C_{ij}) is positive definite.

To choose such coordinates one chooses a one-parameter family of hypersurfaces parameterized by v and foliated by null geodesics. The coordinate along the prescribed geodesics is given by u and γ is given by $(u, 0, 0)$.

In other words, one chooses a local extension of the null geodetic tangent vectorfield $\dfrac{\partial}{\partial u}$ of γ to a null geodetic vector field in a neighborhood of x. Then one chooses $(n-1)$-submanifolds on which the restricted metric is riemannian and allows v to be the parameter labelling these submanifolds.

Let $\Omega \in (0, \infty)$. Consider the map

$$\psi_\Omega : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : (u, v, [y^k]) \mapsto (u, \Omega^2 v, [\Omega y^k]). \tag{2}$$

This map induces the following change of coordinates:

$$\phi^U_\Omega = \mu^{-1} \circ \psi_\Omega \circ \mu : U \to U. \tag{3}$$

(If necessary, to make this well defined, we may need to shrink U so that it does not contain any “holes".) By patching together such coordinate neighborhoods along γ we may think of ϕ_Ω as a diffeomorphism from a tubular neighborhood of γ to a tubular subneighborhood. If we apply this change of coordinates to g, rescale the result by Ω^{-2} and then take the limit as $\Omega \to 0$ we obtain a well defined metric:

$$g_{Pl} = \lim_{\Omega \to 0} \Omega^{-2} (\phi_\Omega^{-1})^* g = \mathrm{d}u\mathrm{d}v + \sum_{i,j=1}^{n-1} C_{ij}(u, 0, 0)\mathrm{d}y^i \mathrm{d}y^j. \tag{4}$$

We call g_{Pl}, together with the tubular neighborhood of γ, the **Penrose limit** of (M, g) along γ. Notice that at $\Omega = 0$, ϕ_Ω is no longer a diffeomorphism.

Proposition 1. *g_{Pl} is defined independently of choice of coordinates putting g in the form* (1).

Proof. Let $(r, s, [x^i])$ be a different choice of coordinates such that

$$g = \mathrm{d}r\mathrm{d}s + \rho \mathrm{d}s^2 + \sum_{i=1}^{n-1} \psi_i \mathrm{d}x^i \mathrm{d}s + \sum_{i,j=1}^{n-1} \Theta_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{5}$$

where $\rho, \psi_i, \Theta_{ij}$ are functions of $(r, s, [x^i])$ and (Θ_{ij}) is positive definite. As both u and r are parameters along the geodesic γ we may as well choose them equal $u = r$. An easy check shows that the change of coordinates matrix must be of the form

$$\begin{pmatrix} \mathrm{d}r \\ \mathrm{d}s \\ \mathrm{d}x^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c^i & e^i_k \end{pmatrix} \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}v \\ \mathrm{d}y^k \end{pmatrix}$$

and that under this

$$\Theta_{ij} e^i_k e^j_l = C_{kl}. \tag{6}$$

In fact c^i must also be zero because the second row in the matrix equation above shows that $s = v + K$, K a constant, and the change of basis matrix for the dual basis to the one-forms above is the inverse transpose:

$$\begin{pmatrix} \partial u \\ \partial v \\ \partial y^i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c^i (e^{-1})^i_k \\ 0 & 0 & (e^{-1})^i_k \end{pmatrix} \begin{pmatrix} \partial r \\ \partial s \\ \partial x^k \end{pmatrix}$$

As e^i_k is nondegenerate we must have $c^i = 0$. Putting this into the Penrose limit metric (4)

$$\begin{aligned} \mathrm{d}r\mathrm{d}s + \sum_{i,j=1}^{n-1} \Theta_{ij}(r,0,0)\mathrm{d}x^i\mathrm{d}x^j &= \mathrm{d}u\mathrm{d}v + \sum_{i,j,k,l=1}^{n-1} \Theta_{ij}(r,0,0)e^i_k e^j_l \mathrm{d}y^k\mathrm{d}y^l \\ &= \mathrm{d}u\mathrm{d}v + \sum_{k,l=1}^{n-1} C_{kl}(u,0,0)\mathrm{d}y^k\mathrm{d}y^l. \qquad \square \end{aligned} \tag{7}$$

In the recent paper, [3] a covariant description of the Penrose limit without reference to the adapted coordinates is given.

A sufficient condition for telling when two Penrose limits will be isometric is the following (the statement of this Theorem appeared in [5] although the proof did not).

Theorem 2 (Covariance of the Penrose limit). *Let (M, g), (M', g') be two lorentzian manifolds. Let γ and γ' be two null geodesics inside M and M', respectively. Let $f : M_\gamma \to M'_{\gamma'}$ be an isometry of tubular neighborhoods of γ and γ' which maps γ onto γ'. Then the Penrose limits of (M, g) and (M', g') along γ and γ', respectively are isometric.*

Proof. Let $(U, \mu = (u, v, [y^k]))$ be a coordinate neighborhood of a point x on γ such that the metric g takes the form (1). Define a coordinate neighborhood $(f(U), \mu' = (u', v', [y'^k]))$ about $f(x)$ by

$$\mu'(f(x)) = \mu(x), \tag{8}$$

so that $u' = u \circ f^{-1}$ is a coordinate along γ'. As $g = f^*g'$, then g' also takes the form of (1) in this neighborhood.

Now consider $f \circ \phi^U_\Omega : U \to U'$. We have

$$\begin{aligned} f \circ \phi^U_\Omega &= f \circ \mu^{-1} \circ \psi_\Omega \circ \mu = f \circ (\mu' \circ f)^{-1} \circ \psi_\Omega \circ (\mu' \circ f) \\ &= \mu'^{-1} \circ \psi_\Omega \circ \mu' \circ f = \phi^{U'}_\Omega \circ f. \end{aligned} \tag{9}$$

Therefore,

$$\begin{aligned} g_{Pl} &= \lim_{\Omega\to 0} \Omega^{-2}(\phi^U_\Omega)^* g = \lim_{\Omega\to 0} \Omega^{-2}(\phi^U_\Omega)^* f^* g' \\ &= \lim_{\Omega\to 0} \Omega^{-2}(f \circ \phi^U_\Omega)^* g' = \lim_{\Omega\to 0} \Omega^{-2}(\phi^{U'}_\Omega \circ f)^* g' \\ &= \lim_{\Omega\to 0} \Omega^{-2} f^* \circ (\phi^{U'}_\Omega)^* g' = f^* g'_{Pl}. \qquad \square \end{aligned} \tag{10}$$

3. Hereditary properties

We say that a property of the metric g is **hereditary** if the Penrose limit g_{Pl} has the same property. For example,

Proposition 3. *Suppose (M, g) is locally symmetric/conformally flat. Then (M_γ, g_{Pl}) is locally symmetric/conformally flat. If (M, g) is Einstein then (M_γ, g_{Pl}) is Ricci flat, in particular it is Einstein.*

Proof. Let ∇_Ω, R_Ω denote the connection and curvature of $g_\Omega := \Omega^{-2}(\phi_\Omega^{-1})^* g$, respectively. As ϕ_Ω is a diffeomorphism if $\nabla R = 0$ then $\nabla_\Omega R_\Omega = 0$ for $\Omega > 0$. By a continuity argument we see that $\nabla_{Pl} R_{Pl} = 0$.

If $\mathrm{Ric}(g) = \lambda g$ then

$$\mathrm{Ric}(g_\Omega) = \mathrm{Ric}(\Omega^{-2}(\phi_\Omega^{-1})^* g) = \mathrm{Ric}((\phi_\Omega^{-1})^* g) = \lambda(\phi_\Omega^{-1})^* g. \tag{11}$$

This gives

$$\mathrm{Ric}(g_\Omega) = \Omega^2 \lambda g_\Omega, \tag{12}$$

and by continuity we see that $\mathrm{Ric}(g_{Pl}) = 0$. □

These hereditary properties can be used to easily compute the Penrose limits of anti de Sitter space *AdS*. Anti de Sitter space is Einstein and conformally flat hence any Penrose limit is Ricci flat and conformally flat and thus flat.

In [5] the case of $AdS \times S$ is considered. It is a symmetric space and is shown to have two non-isometric null geodesics leading to two non-isometric Penrose limits which are flat space and a symmetric plane wave.

Another useful hereditary property is that of geodesic completeness:

Theorem 4. *Suppose (M, g) is a geodesically complete lorentzian manifold. Then the Penrose limit along any null geodesic is geodesically complete.*

Proof. Let $\gamma(t)$ be a geodesic with respect to ∇_{Pl} for $t \in [a, b]$. Without loss we may assume that γ is contained in a normal coordinate neighborhood of some point on γ so that there is a unique geodesic from $\gamma(a)$ to $\gamma(b)$ with respect to ∇_Ω for $\Omega \in [0, 1]$ (which is possible because ∇_Ω varies continuously with respect to Ω and $[0, 1]$ is compact.). Let γ_Ω be the unique geodesic with respect to ∇_Ω between $\gamma(a)$ and $\gamma(b)$. Then $\gamma_\Omega(t)$ may be extended to $(-\infty, \infty)$ as ∇_1 is geodesically complete and ϕ_Ω is a diffeomorphism. Continuity implies that the sequence of geodesics $\gamma(\Omega)$ for $\Omega = \frac{1}{k}$ “converges" to γ in the following sense. Any neighborhood of any point on γ intersects all but a finite number of geodesics of the sequence. Therefore, by continuity of the geodesic equation with respect to Ω, we have that γ may be extended beyond (a, b). □

One last hereditary property, as noted in the introduction, is the following:

Proposition 5. *The dimension of the isometry algebra of g_{Pl} is no less than the dimension of the isometry algebra of g.*

Proof. See [13] or [5]. □

4. Homogeneous spaces and homogeneous structures

In this section, we will give the definitions and results we need in relation to homogeneous spaces.

Definition 6. A connected lorentzian manifold (M, g) is **homogeneous** if its group of isometries acts transitively on M.

When this is the case then M can be written $M = G/H$ where G is a subgroup of isometries and H is a closed subgroup of G.

Definition 7. A homogeneous space $M = G/H$ is **reductive** when there exists a subspace $\mathfrak{m} \cong T_pM \subset \mathfrak{g}$ such that

(1) $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,
(2) $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

It is **symmetric** if it also satisfies

$$[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \tag{13}$$

(In fact, strictly speaking, this is the definition of **weakly reductive**. However, for the rest of this paper we shall assume that H is connected, in which case they are the same thing.)

Definition 8. Let o denote the coset of H in M and fix a frame $u_o : \mathbb{R}^n \to T_oM$ of the frame bundle F. Define the **linear isotropy representation** $\lambda : H \to GL(n, \mathbb{R})$ by

$$\lambda(h) := u_o^{-1} \circ h_* \circ u_o, \tag{14}$$

where $h \in H$, $h_* : T_oM \to T_oM$ denotes the differential of h at o.p

We may study the geometry of a homogeneous space M by studying the space of invariant connections on M. The following theorem gives a gives a description of the space of such connections on a reductive homogeneous space.

Theorem 9. *Let F be the frame bundle of $M = G/H$ a reductive homogeneous space of dimension n with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Then there is a one-to-one correspondence between the set of G-invariant connections in F and the set of linear maps $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{gl}(n, \mathbb{R})$ such that*

$$\Lambda_{\mathfrak{m}}(adh(X)) = ad(\lambda(h))(\Lambda_{\mathfrak{m}}(X)), \tag{15}$$

for $X \in \mathfrak{m}$ and $h \in H$.

The correspondence is given by

$$\omega_{u_o}(\tilde{X}) = \begin{cases} \lambda(X) & if\ X \in \mathfrak{h}, \\ \Lambda_{\mathfrak{m}}(X) & if\ X \in \mathfrak{m}, \end{cases} \tag{16}$$

where ω is the connection one-form, $\tilde{X}$ is the natural lift of $X \in \mathfrak{g}$ to F and λ is not only as above $H \to GL(n, \mathbb{R})$ but also the induced Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{gl}(n, \mathbb{R})$.

Proof. See chapter X, Theorem 2.1 in [15]. □

Definition 10. The connection obtained by taking $\Lambda_{\mathfrak{m}} = 0$ is called the **canonical connection**.

The canonical connection can also be described in the following way. Let θ be the left-invariant Maurer–Cartan form of G

$$\theta_g(X) := (L_g)^*(X), \tag{17}$$

where L_g denotes left multiplication by g and $*$ denotes differentiation. Let $\sigma : U \to G$ be a local coset representative. Then the pull back of θ by σ splits as

$$\sigma^*(\theta) = \theta^{\mathfrak{h}} + \theta^{\mathfrak{m}}, \tag{18}$$

where $\theta_x^{\mathfrak{h}}(X) \in \mathfrak{h}$, $\theta_x^{\mathfrak{m}}(X) \in \mathfrak{m}$. The one-form $\theta^{\mathfrak{h}}$ defines the connection one-form for the canonical connection.

The geodesics of the canonical connection are curves $\gamma(t)$ of the form

$$\exp(tX), \quad t \in \mathbb{R}, X \in \mathfrak{g}. \tag{19}$$

If (M, g) is symmetric then the canonical connection coincides with the Levi–Civita connection.

Theorem 11. *The canonical connection of a reductive homogeneous space is complete.*

Proof. See chapter X, Corollary 2.5 in [15]. □

The following theorem of Ambrose–Singer [1] shows the importance of the canonical connection as a tool for expressing the algebraic condition of reductive homogeneity as the existence of a solution to a system of differential equations.

Theorem 12 ([1,17,12]). *Let (M, g) be a reductive lorentzian homogeneous manifold with Levi–Civita connection ∇. Then there exists a $(2,1)$ tensor T defining a metric connection $\tilde{\nabla} := \nabla - T$ with curvature R such that $\tilde{\nabla}T = \tilde{\nabla}R = 0$.*

Proof. Write $M = G/H$, with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let $\tilde{\nabla}$ be the canonical connection of M. Let $T = \nabla - \tilde{\nabla}$. As G acts by isometries, ∇ is also G-invariant. Hence T and R are G-invariant. Therefore, see [15], they are both parallel with respect to $\tilde{\nabla}$. □

(The first version of this Theorem for riemannian signature appeared in [1]. This was re-interpreted in terms of the canonical connection in [17] and extended to the pseudo-riemannian case in [12].)

Remarks:.

(1) T is not necessarily the torsion of $\tilde{\nabla}$ (and not necessarily skew-symmetric in it lower indices.) If $\tilde{\Gamma}^i_{jk}$ are the Christofel symbols of $\tilde{\nabla}$ and Γ^i_{jk} of ∇ and τ is the torsion of $\tilde{\nabla}$ then

$$\tau^i_{jk} = \tilde{\Gamma}^i_{jk} - \tilde{\Gamma}^i_{kj} = \tilde{\Gamma}^i_{jk} - \Gamma^i_{jk} + \Gamma^i_{kj} - \tilde{\Gamma}^i_{kj} = -T^i_{jk} + T^i_{kj}. \tag{20}$$

i.e. τ is the skew-symmetrization of T. In fact $\tau(X, Y)|_{\mathfrak{m}} = -[X, Y]_{\mathfrak{m}}$, the component of $[X, Y]$ lying in $\mathfrak{m}$ where $X, Y \in \mathfrak{m}$ (see chapter X, Theorem 2.6 in [15].) Also the restriction of T to $\mathfrak{m}$ is given by

$$T(X, Y)|_{\mathfrak{m}} = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y), \tag{21}$$

where U is the symmetric bilinear mapping of $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$ defined by

$$2g_o(U(X, Y), Z) = g_o(X, [Z, Y]_{\mathfrak{m}}) + g_o([Z, X]_{\mathfrak{m}}, Y), \tag{22}$$

where $X, Y, Z \in \mathfrak{m}$ (see chapter X, Theorem 3.3 in [15].)

(2) If (M, g) is symmetric then $\nabla R = 0$ and we can take $T = 0$.

Such a T-tensor is called a **homogeneous structure**. A given homogeneous manifold M may have many different homogeneous structures. Each corresponding to a different choice of groups G and H. For example, the 7-sphere $S^7 = SO(8)/SO(7) = Spin(7)/G_2 = Sp(2)/Sp(1)$. (For a review of Penrose limits from the point of view of homogeneous structures see [10].)

Definition 13. (M, g) is called **naturally reductive** if there exists a homogeneous structure T with $U = 0$, i.e. if $\tau = T$.

While reductivity is a property of the isotropy representation, natural reductivity is also a property of the metric.

Proposition 14. *Let (M, g) be naturally reductive. Then the geodesics of the Levi–Civita connection coincide with the geodesics of the canonical connection.*

Proof. Let ∇ denote the Levi–Civita connection and $\tilde{\nabla}$ the canonical connection corresponding to the homogeneous structure T with $U = 0$. Then T will be skew-symmetric in its lower indices and consequently

$$\nabla_X X = \tilde{\nabla}_X X + T(X, X) = \tilde{\nabla}_X X. \tag{23}$$

Hence the geodesic equations for the two connections are the same. □

We may rewrite Theorem 12 in terms of the torsion of $\tilde{\nabla}$ instead of T and then include a converse:

Theorem 15. *Let (M, g) be a connected, simply connected, lorentzian manifold. Then (M, g) is reductive homogeneous if and only if there exists a complete affine metric connection $\tilde{\nabla}$ with torsion τ and curvature R such that $\tilde{\nabla}\tau = \tilde{\nabla}R = 0$.*

Proof. See chapter X, Theorems 2.6–2.8 in [15]. □

Theorems 12 and 15 above describe the reductive homogeneity in terms of a metric connection on the tangent bundle. In fact, we can describe the Killing vectors of an arbitrary pseudo-riemannian manifold as parallel vector fields of a covariant derivative on an extended bundle.

Let X be a vector field on a lorentzian manifold (M, g). Let $A_X : Y \mapsto -\nabla_Y X$. Then X is a Killing vector if and only if A_X is skew-symmetric with respect to g. As a consequence of the Killing identity we also have the equation:

$$\nabla_X A_\zeta = -R(X, \zeta)$$

Now consider the bundle $\mathcal{E} = TM \oplus \mathfrak{so}(TM)$. If we define a covariant derivative D on $\mathcal{E}$ by

$$D_X(\zeta, A) := (\nabla_X \zeta + A(X), \nabla_X A + R(X, \zeta)).$$

Then the parallel sections of $\mathcal{E}$ with respect to D are precisely the Killing vectors of g. Thus, a Killing vector is completely determined by

$$(\zeta(p), A_\zeta(p))$$

at any point p and by parallel translation by the covariant derivative D.

Finally we make the

Definition 16. A geodesic γ is called homogeneous if it is the orbit of a 1-parameter subgroup of isometries.

Note: on a riemannian space this definition is equivalent to writing the geodesic in the form $\gamma(t) = \exp(tX)_o$ for some $X \in \mathfrak{g}$ (see [20].) However, if the geodesic $\gamma(t)$ is null one may have to change its parameterization in order to write it in the form $\exp(sX)_o$. We call a geodesic of the form $\gamma(t) = \exp(tX)_o$ an **absolutely homogeneous geodesic**. Also notice that a homogeneous geodesic is not necessarily a geodesic of the canonical connection as a geodesic of the canonical connection is of the form $\exp(tX)$ with $X \in \mathfrak{m}$. We shall call a homogeneous geodesic which is a geodesic of the canonical connection a **canonical homogeneous geodesic**.

These will be the geodesics of interest when deciding whether a Penrose limit is homogeneous or not. A useful criteria for distinguishing homogeneous geodesics is the following,

Proposition 17. *Suppose M is a lorentzian reductive homogeneous space. The geodesic* $\gamma(t)$ with $\gamma(0) = o$ *and* $\gamma'(0) = X \in \mathfrak{g}$ *is a homogeneous geodesic if and only if*

$$B(X_{\mathfrak{m}}, [Z, X]_{\mathfrak{m}}) = \lambda B(X_{\mathfrak{m}}, Z_{\mathfrak{m}}) \tag{24}$$

for all $Z \in \mathfrak{g}$ *and some* $\lambda \in \mathbb{R}$. *It is absolutely homogeneous if and only if* $\lambda = 0$.

Proof. This is a slight generalization of the proof given in [20]. □

Definition 18. A vector $X \in \mathfrak{g}$ which satisfies (24) is called a **geodesic vector**.

Note: by putting $Z = X$ in (24), we see that if B has riemannian signature then we must have $\lambda = 0$.

Remark:. Suppose γ is a geodesic parameterized by u. If γ is homogeneous then there is a Killing vector ζ such that $\zeta_p = \gamma'_p$ at all points $p \in \gamma$. But the geodesic vector field $\frac{\partial}{\partial u}$ is not necessarily a Killing vector field. If the Killing vector field ζ is given by $\frac{\partial}{\partial t^1}$, then t^1 may not be part of a **twist–free coordinate system**; that is a coordinate system $(t^1, \ldots, t^n)$, in which we can write the metric in the form $g_{ij}dt^i dt^j$ such that $d(g_{1i}\, dt^i) = 0$. In particular, if γ is a null homogeneous geodesic then we may not be able to write g in the form of (1) with $\frac{\partial}{\partial u}$ a Killing vector.

Proposition 19. *A geodesic* γ *is homogeneous if and only if there exists a solution* (γ, A) *to the Killing transport equations with* $A(\gamma') = 0$.

5. Penrose limits along homogeneous geodesics

In this section, we will give three Theorems which give sufficient conditions for homogeniety to be hereditary.

Theorem 20. *The Penrose limit along a null geodesic* $\frac{\partial}{\partial u}$ *which is a Killing vector is flat.*

Proof.

$$0 = \mathfrak{L}_{\frac{\partial}{\partial u}} g = \mathrm{d}(i_{\frac{\partial}{\partial u}} g) + i_{\frac{\partial}{\partial u}} \mathrm{d}g = \mathrm{d}(\mathrm{d}v) + \frac{\partial \alpha}{\partial u}\mathrm{d}v^2 + \frac{\partial \beta_i}{\partial u}\mathrm{d}v\mathrm{d}y^i + \frac{\partial C_{ij}}{\partial u}\mathrm{d}y^i\mathrm{d}y^j.$$

Therefore, C is independent of u and hence g_{Pl} is flat. □

Theorem 21. *The Penrose limit of a lorentzian metric along a homogeneous geodesic* γ *is homogeneous.*

Proof. On a plane wave $\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v + C_{ij}\mathrm{d}y^i\mathrm{d}y^j$, we have the Killing vectors

$$\frac{\partial}{\partial v}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{n-1}}$$

which are independent at each point p. So to prove g_{Pl} is locally homogeneous it is enough to show it has a Killing vector which agrees with $\frac{\partial}{\partial u} = \gamma'$ at p. Suppose that ζ is a Killing vector such that $\zeta|_\gamma = f\frac{\partial}{\partial u}|_\gamma$. Then $\zeta|_\gamma$ is generated by Killing transport of $(\zeta(p), A_\zeta(p))$ along γ. Now by definition,

$$(A_\zeta f\gamma')|_\gamma = (A_\zeta\zeta)|_\gamma = 0,$$

where by $|_\gamma$ we mean restriction to $\gamma \in M$, not restriction of the tangent bundle. Therefore, if we write A_ζ in components:

$$A_\zeta = \sum_{i,j}(A_\zeta)_i^j dx^i \otimes \frac{\partial}{\partial x^j},$$

we see that

$$(A_\zeta)_u^{y^i}|_\gamma = 0.$$

Also, as ζ is a Killing vector, we have

$$g(A_\zeta X, Y) = -g(X, A_\zeta Y).$$

Therefore,

$$(A_\zeta)_{y^i}^v|_\gamma = (A_\zeta)_u^v = 0.$$

Note that the restriction to the geodesic is not necessary for $(A_\zeta)_u^v$.

Now consider the pull-back of the Killing transport covariant derivative under the Penrose limit map ϕ_Ω;

$$(\phi_\Omega^{-1})^* D_\zeta(X) = (\phi_\Omega^{-1})^*\nabla_\zeta(X) - (\phi_\Omega^{-1})^* A_\zeta(X).$$

The components of A_ζ scale under the Penrose limit map in the following way:

$$(A_\zeta)_u^{y^i}(u, v, y^k) \mapsto \Omega^{-1}(A_\zeta)_u^{y^i}(u, \Omega^2 v, \Omega y^k)$$
$$(A_\zeta)_{y^i}^v(u, v, y^k) \mapsto \Omega^{-1}(A_\zeta)_{y^i}^v(u, \Omega^2 v, \Omega y^k)$$

and other components which either stay constant or tend to zero as $\Omega \to 0$. Taking the limit as $\Omega \to 0$ by using L'Hopital's rule, we have

$$(A_\zeta)_u^{y^i}(u, v, y^k) \mapsto y^j \tfrac{\partial}{\partial y^j}(A_\zeta)_u^{y^i}(u, 0, 0)$$
$$(A_\zeta)_{y^i}^v(u, v, y^k) \mapsto y^j \tfrac{\partial}{\partial y^j}(A_\zeta)_{y^i}^v(u, 0, 0)$$

Therefore,

$$(D_{Pl})_\zeta(X)(u, v, y) := \lim_{\Omega\to 0}[D_\zeta(X)(u, 0, 0)]$$

is well-defined and along with

$$(D_{Pl})_\zeta(A) := (\nabla_{Pl})_\zeta A - R_{Pl}(\zeta, X),$$

defines a Killing transport covariant derivative on along γ with respect to g_{Pl}. Therefore, parallel translation by D_{Pl} along γ generates the remaining Killing vector needed.

Corollary 22. *If γ is an (absolutely) homogeneous geodesic of g then it is also an (absolutely) homogeneous geodesic of the Penrose limit of (g, γ).*

When (M, g) is a reductive homogeneous manifold we can use the same strategy as above to construct a homogeneous structure on the Penrose limit:

Proposition 23. *The Penrose limit of a reductive lorentzian homogeneous manifold along a canonical homogeneous geodesic is locally reductive homogeneous.*

Proof. Let (M, g) be a reductive homogeneous space with a null homogeneous geodesic γ. From the Ambrose–Singer Theorem we have a connection $\tilde{\nabla}$ such that $\tilde{\nabla}T = \tilde{\nabla}R = 0$. Let M_γ be a tubular neighborhood of γ as above and consider $\phi_\Omega(M_\gamma)$. Now ϕ_Ω is a diffeomorphism for $\Omega \neq 0$ so $\phi_\Omega(M_\gamma)$ is reductive homogeneous for $\Omega > 0$. This defines the metric connection

$$\tilde{\nabla}_\Omega := (\phi_\Omega^{-1})^* \tilde{\nabla} = (\phi_\Omega^{-1})^* \nabla - (\phi_\Omega^{-1})^* T. \tag{25}$$

γ is a homogeneous geodesic of (M, g) so T is a tensor of type $(2, 1)$

$$T = \nabla - \tilde{\nabla} = \sum_{i,j,k=1}^{n+1} T_{ij}^k \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes \frac{\partial}{\partial x^k}. \tag{26}$$

Under the Penrose limit map ϕ_Ω the coefficients scale in the following way

$$\begin{aligned} T_{uy^i}^v &\mapsto \Omega^{-1} T_{uy^i}^v(u, \Omega^2 v, \Omega y^k) \\ T_{uu}^v &\mapsto \Omega^{-2} T_{uu}^v(u, \Omega^2 v, \Omega y^k) \\ T_{uu}^{y^i} &\mapsto \Omega^{-1} T_{uu}^{y^i}(u, \Omega^2 v, \Omega y^k), \end{aligned}$$

and terms which either remain the same or tend to 0 in the limit $\Omega \to 0$. Suppose that γ is a canonical homogeneous geodesic. Then there is a Killing vector ζ such that $\zeta|_\gamma = f \frac{\partial}{\partial u}|_\gamma$. Then $\zeta|_\gamma$ is generated by parallel transport by the canonical connection of $\zeta(p)$ along γ. Now by definition,

$$(\nabla_{\gamma'} \gamma')|_\gamma = 0 \quad \text{and} \quad (\tilde{\nabla}_\zeta \zeta)|_\gamma = 0$$

where by $|_\gamma$ we mean restriction to $\gamma \in M$ not restriction of the tangent bundle. Thus,

$$0 = (\tilde{\nabla}_{f\gamma} f\gamma)|_\gamma = (\nabla_{f\gamma'} f\gamma')|_\gamma - T(f\gamma', f\gamma')|_\gamma = f df(\gamma')\gamma'|_\gamma - f^2 T(\gamma', \gamma')|_\gamma,$$

and therefore,

$$T_{uu}^{y^i}|_\gamma = 0.$$

Also, as $\tilde{\nabla}$ is metric we have

$$\begin{aligned} 0 &= (\tilde{\nabla}_W g)(X, Y) \\ &= (\nabla_W g)(X, Y) + g(T_W X, Y) + g(X, T_W Y) \\ &= g(T_W X, Y) + g(X, T_W Y), \end{aligned} \tag{27}$$

as ∇ is metric. Hence using (1) we see that

$$T_{uu}^v = 0 \quad \text{and} \quad T_{uy^i}^v|_\gamma = 0, \tag{28}$$

The Levi–Civita connection of the Penrose limit along γ, ∇_{Pl}, is equal to

$$\lim_{\Omega \to 0} (\phi_\Omega^{-1})^* \nabla. \tag{29}$$

Now, using L'Hopital's rule to take the limit of $(\phi_\Omega^{-1})^* T$ as Ω tends to 0 we find,

$$T^v_{uy^i} \mapsto y^j \left(\frac{\partial}{\partial y^j} T^v_{uy^i}\right)(u, 0, 0)$$
$$T^{y^i}_{uu} \mapsto y^j \left(\frac{\partial}{\partial y^j} T^{y^i}_{uu}\right)(u, 0, 0),$$

So we can make the definition $T_{Pl} := \lim_{\Omega\to 0}(\phi_\Omega^{-1})^* T$, whence by (25), the limit $\tilde\nabla_{Pl} := \lim_{\Omega\to 0}\tilde\nabla_\Omega$ is well defined. Now

$$\{\tilde\nabla_\Omega g_\Omega | \Omega \in [0, 1]\}$$

is a continuous path in the space of tensors of type $(3, 0)$ on γ. Therefore, continuity shows $\tilde\nabla_{Pl} g_{Pl} = 0$. Similarly

$$\tilde\nabla_{Pl} g_{Pl} = \tilde\nabla_{Pl} T_{Pl} = \tilde\nabla_{Pl} R_{Pl} = 0. \tag{30}$$

Applying Theorem 15 gives the result. □

Corollary 24. *A homogeneous structure T has a well–defined Penrose limit along a null geodesic $\gamma(t)$ if and only if $\gamma(t)$ can be re–parameterized to a geodesic of the canonical connection with respect to T.*

Proof. T has a well–defined limit if and only if $T^{y^i}_{uu}|_\gamma = T^v_{uu}|_\gamma = 0$. The proof of Proposition 23 shows that this is the case if and only if γ can be re–parameterized to a geodesic of the canonical connection. □

Corollary 25. *Suppose that (M, g) is a naturally reductive space. Then for any null canonical geodesic γ, the Penrose limit of (M, g, γ) is homogeneous.*

6. Homogeneous plane waves

We can learn more about the hereditary properties of homogeneity by studying the space of homogeneous plane waves. In [7], Blau and O'Loughlin have classified all homogeneous plane waves into two classes. The first class consists of complete metrics and the second class incomplete metrics:

Theorem 26 (Blau–O'Loughlin [7]). *There are two classes of homogeneous plane waves*:

(1) $g = 2\mathrm{d}x^+\mathrm{d}x^- + (e^{x^+f} A_0 e^{-x^+f})_{ij} z^i z^j (\mathrm{d}x^+)^2 + \sum_i (\mathrm{d}z^i)^2$. *Complete metrics.*

(2) $g = 2\mathrm{d}x^+\mathrm{d}x^- + (e^{f\log x^+} A_0 e^{-f\log x^+})_{ij} z^i z^j \dfrac{(\mathrm{d}x^+)^2}{(x^+)^2} + \sum_i (\mathrm{d}z^i)^2$. *Incomplete metrics (singularity along x^+).*

The isometry algebra of the generic homogeneous plane wave is given by:

$$[e_i, Y_j] = \delta_{ij} Z, \quad [e_i, X] = -Y_i,$$
$$[Y_i, Y_j] = 2 f_{ij} Z, \quad [X, Z] = aZ$$
$$[X, Y_i] = (a\delta_{ij} + 2 f_{ij}) Y_j + ((A_0)_{ij} - a f_{ij} - f_{ik} f_{kj}) e_j$$

Here $(A_0)_{ij}$ is symmetric and f_{ij} skew–symmetric. The isotropy is generated by the e_i's. From this it is clear that homogeneous plane waves are reductive. The non–singular plane waves have an isometry algebra with $a = 0$, while the singular plane waves have an algebra with $a = 1$. By calculating the homogeneous structure associated to these reductive splittings we see that the non–singular plane waves are naturally reductive, while the singular plane waves are not.

Contained in the class of naturally reductive plane waves are the symmetric plane waves, also called the Cahen–Wallach spaces (see [9] for the original paper or [11].) These are given by taking $f_{ij} = 0$ in (1) of Theorem 26 and can be diagonalised to the form:

$$g = 2\mathrm{d}x^+\mathrm{d}x^- + \sum_i A_i(z^i)^2(\mathrm{d}x^+)^2 + \sum_i(\mathrm{d}z^i)^2$$

with A_i constant.

Combining this classification with Corollary 22 we obtain the

Theorem 27. *The Penrose limit of a lorentzian space along an absolutely homogeneous geodesic is a naturally reductive plane wave.*

Also we have the

Proposition 28. *The Penrose limit of a geodesically complete lorentzian metric g along a homogeneous geodesic is naturally reductive homogeneous.*

Proof. If g is geodesically complete then the Penrose limit is complete. The classification of homogeneous plane-waves shows that a complete homogeneous plane-wave is naturally reductive. □

7. Examples

In this section, we will give some examples to show that the above Theorems cannot be strengthened any further.

First we will show that the converse to Theorem 21 is not true, i.e. we give an example which shows that the Penrose limit of a non–homogeneous geodesic in a non–homogeneous space may be homogeneous. Consider the metric

$$g = 2\mathrm{d}u\mathrm{d}v + u\mathrm{d}v^2 + \sqrt{u}\sum_i(\mathrm{d}x^i)^2. \tag{31}$$

This is an incomplete and non–homogeneous metric with no Killing vector in the ∂_u direction. Therefore, the null geodesic given by ∂_u is not homogeneous. However the Penrose limit of (g, ∂_u) is given by

$$2\mathrm{d}u\mathrm{d}v + \sqrt{u}\sum_i(\mathrm{d}x^i)^2. \tag{32}$$

This is a reductive plane wave [23]:

Next we will consider non–homogeneous geodesics in a homogeneous space. In [24] Patricot calculated the Penrose limits of the **Kaigorodov space** K_{n+3} which is $\mathbb{R}^{n+3}$ together with the metric:

$$g_{n+3} = e^{-2nL\rho}\mathrm{d}x^2 + e^{4L\rho}(2\mathrm{d}x\mathrm{d}t + \sum_{i=1}^{n}(\mathrm{d}y^i)^2) + \mathrm{d}\rho^2\ ,$$

where $L = \frac{1}{2}\sqrt{-\frac{\Lambda}{n+2}}$.

This is a homogeneous space whose isometries are generated by

$$\boldsymbol{K}_{(0)} = \frac{\partial}{\partial t}, \quad \boldsymbol{K}_{(x)} = \frac{\partial}{\partial x}, \quad \boldsymbol{K}_{(i)} = \frac{\partial}{\partial y^i}, \quad \boldsymbol{L}_i = x\frac{\partial}{\partial y^i} - y^i\frac{\partial}{\partial t}, \quad \boldsymbol{L}_{ij} = y^i\frac{\partial}{\partial y^j} - y^j\frac{\partial}{\partial y^i},$$

$$\boldsymbol{J} = \frac{\partial}{\partial \rho} - at\frac{\partial}{\partial t} - bt\frac{\partial}{\partial x} - cy^i\frac{\partial}{\partial y^i},$$

Here $a = (n+4)L$, $b = -nL$ and $c = 2L$.

$[,]$	$\boldsymbol{L}_r$	$\boldsymbol{L}_{uv}$	$\boldsymbol{K}_0$	$\boldsymbol{K}_x$	$\boldsymbol{K}_i$	$\boldsymbol{J}$
$\boldsymbol{L}_r$	0	$\delta_{ur}\boldsymbol{K}_v - \delta_{vr}\boldsymbol{K}_u$	0	$-\boldsymbol{K}_r$	$\delta_{ri}\boldsymbol{K}_0$	$(c-a)\boldsymbol{L}_r$
$\boldsymbol{L}_{st}$	$-\delta_{sr}\boldsymbol{L}_t + \delta_{tr}\boldsymbol{L}_s$	$\boldsymbol{L}_{stuv}$	0	0	$\delta_{ti}\boldsymbol{K}_s - \delta_{is}\boldsymbol{K}_t$	0
$\boldsymbol{K}_0$	0	0	0	0	0	$-a\boldsymbol{K}_0$
$\boldsymbol{K}_x$	$\boldsymbol{K}_r$	0	0	0	0	$-b\boldsymbol{K}_x$
$\boldsymbol{K}_i$	$-\delta_{ri}\boldsymbol{K}_0$	$\delta_{iu}\boldsymbol{K}_v - \delta_{iv}\boldsymbol{K}_u$	0	0	0	$-c\boldsymbol{K}_i$
$\boldsymbol{J}$	$(a-c)\boldsymbol{L}_r$	0	$a\boldsymbol{K}_0$	$b\boldsymbol{K}_x$	$c\boldsymbol{K}_i$	0

where $\boldsymbol{L}_{stuv} = -\delta_{su}\boldsymbol{L}_{tv} + \delta_{tu}\boldsymbol{L}_{sv} - \delta_{jl}\boldsymbol{L}_{ik} + \delta_{il}\boldsymbol{L}_{jk}$. This isometry algebra is the semidirect product of an extended Heisenberg algebra and $\mathfrak{so}(n)$ and has dimension $(2n+3) + \frac{1}{2}n(n+1)$. The homogeneous space as given by the full isometry group is non–reductive. However, the transitive subgroup which is generated by $\boldsymbol{K}_0$, $\boldsymbol{K}_x$, $\boldsymbol{K}_i$ and $\boldsymbol{J}$ gives K_{n+3} as a reductive homogeneous space.

K_{n+3} has 3 non-isometric Penrose limits. The first is along a Killing vector and is thus flat. The second along a homogeneous geodesic is a reductive homogeneous plane wave and the third is along a non–homogeneous geodesic is non–homogeneous.

Patricot also considered the space $K_{n+3} \times S^d$ where S^d is sphere with round metric which is again a non-reductive homogeneous space. It has four non-isometric Penrose limits. Three along geodesics which are constant on the sphere and hence have the same Penrose limits as K_{n+3}; the flat metric and a reductive plane wave and a non–homogeneous metric. The fourth Penrose limit is along a non-homogeneous geodesic which wraps around the sphere and K_{n+3} and is also a non-homogeneous plane wave.

Thus, the Penrose limit of a non-reductive homogeneous space along a non-homogeneous geodesic is not necessarily homogeneous. The next example will illustrate the existence of non–canonical homogeneous geodesics and will show that the Penrose limit of a homogeneous structure along such a curve will blow up.

B. Komrakov Jnr has compiled a complete classification of four-dimensional pseudo-riemannian homogeneous spaces [16]. In his paper he considers the isotropy representation $\rho : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ of a homogeneous space G/H and classifies first all the complex forms and then the real forms of the subalgebra $(\rho(\mathfrak{h}))^{\mathbb{C}} \subset \mathfrak{so}(4, \mathbb{C})$. The result is a list of the possible Lie algebras $\mathfrak{g}$ and chosen subalgebras $\mathfrak{h}$ and the associated isotropy representation given as a matrix.

We can then use the Maurer–Cartan form to recover the metric from B. We summarize below some of the properties of this classification:

- Number of isotropy representations admitting riemannian metrics: 6
- Number of isotropy representations admitting lorentzian metrics:14
- Number of isotropy representations admitting metrics of (2,2) signature: 30

(There is some overlap in these cases where a representation admits metrics of different signatures.)

- Number of symmetric/reductive algebras admitting a riemannian metric: 21/29
- Number of symmetric/reductive/nonreductive algebras admitting a lorentzian metric: 35/64/6
- Number of symmetric/reductive/non-reductive algebras admitting a metric of (2,2) signature: 57/123/9

By studying Komrakov's list we see that there does not exist a four–dimensional lorentzian homogeneous space with a non–canonical homogeneous geodesic. However there do exist 5–dimensional examples as we will now show. Consider the algebra (Komrakov number $1.1^2.11$ extended by a central element.)

$[,]$	$\boldsymbol{e}_1$	$\boldsymbol{u}_1$	$\boldsymbol{u}_2$	$\boldsymbol{u}_3$	$\boldsymbol{u}_4$	$\boldsymbol{u}_5$
$\boldsymbol{e}_1$	0	$\boldsymbol{u}_3$	$\frac{1}{2}\boldsymbol{u}_4$	$-\boldsymbol{u}_1$	$-\frac{1}{2}\boldsymbol{u}_2$	0
$\boldsymbol{u}_1$	$-\boldsymbol{u}_3$	0	$\boldsymbol{u}_2$	$-4\boldsymbol{e}_1$	$-\boldsymbol{u}_4$	0
$\boldsymbol{u}_2$	$-\frac{1}{2}\boldsymbol{u}_4$	$\boldsymbol{u}_2$	0	$-\boldsymbol{u}_4$	0	0
$\boldsymbol{u}_3$	$\boldsymbol{u}_1$	$4\boldsymbol{e}_1$	$\boldsymbol{u}_4$	0	$\boldsymbol{u}_2$	0
$\boldsymbol{u}_4$	$\frac{1}{2}\boldsymbol{u}_2$	$\boldsymbol{u}_4$	0	$-\boldsymbol{u}_2$	0	0
$\boldsymbol{u}_5$	0	0	0	0	0	0

This defines a reductive homogeneous space G/H with $\mathfrak{m}$ the span of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$ and $\mathfrak{h}$ spanned by $\boldsymbol{e}_1$. The corresponding isotropy representation is skew–symmetric with respect to the bilinear form B:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{33}$$

To determine the induced metric we make a choice of local coset representative

$$\sigma = \exp(x_1\boldsymbol{u}_1)\exp(x_2\boldsymbol{u}_2)\exp(x_3\boldsymbol{u}_3)\exp(x_4\boldsymbol{u}_4)\exp(x_5\boldsymbol{u}_5) : M \to G, \tag{34}$$

and calculate the Maurer–Cartan form

$$\begin{aligned}\sigma^{-1}\mathrm{d}\sigma = {} & \cosh(2x_3)\mathrm{d}x_1\boldsymbol{u}_1 \\ & + (x_4\sinh(2x_3)\mathrm{d}x_1 + x_2\cosh(x_3)\mathrm{d}x_1 + \cosh(x_3)\mathrm{d}x_2 + x_4\mathrm{d}x_3)\boldsymbol{u}_2 + \mathrm{d}x_3\boldsymbol{u}_3 \\ & + (-x_4\cosh(2x_3)\mathrm{d}x_1 \\ & - 2\sinh(2x_3)\mathrm{d}x_1 - x_3\sinh(x_3)\mathrm{d}x_1 - \sinh(x_3)\mathrm{d}x_2 + \mathrm{d}x_4)\boldsymbol{u}_4 + \mathrm{d}x_5\boldsymbol{u}_5.\end{aligned}$$

The metric is given by $B((\sigma^{-1}\mathrm{d}\sigma)_{\mathfrak{m}}, (\sigma^{-1}\mathrm{d}\sigma)_{\mathfrak{m}})$. The non–zero components of the homogeneous structure T restricted to the subspace $\mathfrak{m}$ are given by:

$$\begin{aligned} 1 &= T_{414} = T_{221} = T_{243} = T_{244} = T_{423} = T_{424}, \\ -1 &= T_{212} = T_{234} = T_{432} = T_{441}. \end{aligned}$$

Now consider the vector $U = \boldsymbol{u}_2 + \frac{1}{\sqrt{2}}\boldsymbol{u}_3 + \sqrt{\frac{3}{2}}\boldsymbol{u}_5 + \sqrt{2}\boldsymbol{e}_1$. This is an absolutely geodetic vector and hence generates a homogeneous geodesic. However this geodesic is not a geodesic of the canonical connection and thus the Penrose limit of the homogeneous structure along $\exp(tU)(p)$ will blow up:

$$T(U,U)|_{\mathfrak{m}} = \boldsymbol{u}_1 - \frac{1}{2}\boldsymbol{u}_4 \overset{Pl}{\rightarrow} \infty. \tag{35}$$

However, the algebra $\mathfrak{g}$ has the following transitive subalgebra:

$[,]$	U	$\boldsymbol{u}_1$	$\boldsymbol{u}_2$	$\boldsymbol{u}_4$	$\boldsymbol{u}_5$
U	0	$2U - 3\boldsymbol{u}_2$	$\sqrt{2}\boldsymbol{u}_4$	0	0
$\boldsymbol{u}_1$	$-2U + 3\boldsymbol{u}_2$	0	$\boldsymbol{u}_2$	$-\boldsymbol{u}_4$	0
$\boldsymbol{u}_2$	$\sqrt{2}\boldsymbol{u}_4$	$-\boldsymbol{u}_2$	0	0	0
$\boldsymbol{u}_4$	0	$\boldsymbol{u}_4$	0	0	0
$\boldsymbol{u}_5$	0	0	0	0	0

The non–zero components of the homogeneous structure are given by

$$-1 = T_{U12} = T_{441}, \quad 1 = T_{221} = T_{21U} = T_{212} = T_{414}, \quad \frac{1}{\sqrt{2}} = T_{U24} = T_{24U}.$$

This homogeneous structure does not blow up under the Penrose limit along U. It is not clear that every homogeneous geodesic is canonical with respect to some reductive decomposition as in this case. However, this is true for all the null homogeneous geodesics of four–dimensional lorentzian homogeneous spaces.

8. The existence of homogeneous geodesics

Finally we would like to make a remark on the existence of null homogeneous geodesics. The following Theorem has been proven in [19,18].

Theorem 29 (Kowalski–Szenthe). *Every homogeneous riemannian manifold admits at least one homogeneous geodesic through every point.*

Since every homogenous riemannian manifold is reductive (see [26]) it appears that this Theorem is also true in the case of reductive lorentzian manifolds.

Proposition 30 (*Every reductive homogeneous lorentzian manifold admits at least one homogeneous geodesic through every point*).

In fact all lorentzian homogeneous examples known to the author (and this includes all four–dimensional homogeneous spaces appearing on Komrakov's list,) contain at least one null homogeneous geodesic although not all of them contain an absolutely homogeneous one as the following example (Komrakov number 1.1^2) shows: together with the bilinear form

$[,]$	$\boldsymbol{e}_1$	$\boldsymbol{u}_1$	$\boldsymbol{u}_2$	$\boldsymbol{u}_3$	$\boldsymbol{u}_4$
$\boldsymbol{e}_1$	0	$\boldsymbol{u}_3$	0	$-\boldsymbol{u}_1$	0
$\boldsymbol{u}_1$	$-\boldsymbol{u}_3$	0	0	$-\boldsymbol{u}_2$	$\boldsymbol{u}_1$
$\boldsymbol{u}_2$	0	0	0	0	$2\boldsymbol{u}_2$
$\boldsymbol{u}_3$	$\boldsymbol{u}_1$	$\boldsymbol{u}_2$	0	0	$\boldsymbol{u}_3$
$\boldsymbol{u}_4$	0	$-\boldsymbol{u}_1$	$-2\boldsymbol{u}_2$	$-\boldsymbol{u}_3$	0

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This is a reductive algebra and so using Proposition 17 it can be shown that the homogeneous space derived from this algebra and bilinear form has no null absolutely homogeneous geodesics. (This is in fact effectively the only four-dimensional lorentzian homogeneous space without any null absolutely homogeneous geodesics.) However it does have a family of null homogeneous geodesics.

$$U = A\boldsymbol{u}_4 \pm A\boldsymbol{u}_2 + B\boldsymbol{e}_1 \quad \text{and} \quad \lambda = -2A \quad \text{with} \quad A, B \in \mathbb{R}$$

or

$$U = A\boldsymbol{u}_2 + B\boldsymbol{u}_3 + C\boldsymbol{u}_4 \quad \text{and} \quad \lambda = -C \quad \text{with} \quad A^2 + B^2 = C^2, \quad A, B, C \in \mathbb{R}.$$

To the author's knowledge there are no known results about the existence of homogeneous geodesics in the nonreductive case.

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